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MAXIMIZING STATIONARY UTILITY IN A CONSTANT TECHNOLOGY

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MAXIMIZING STATIONARY UTILITY IN A CONSTANT TECHNOLOGY*

by

Richard Beals** and Tjalling C. Koopmans***

1. Introduction

This paper is concerned with a problem in the optimal control of a nonstochastic process over time. It can also be looked on as a problem in convex programming in a space of infinite sequences of real numbers. Because the problem arose in the theory of optimal economic growth, the exposition will use some economic terminology.

The literature on optimal economic growth contains several papers**** in which a utility function of the form

$$(1) \qquad U(x_1, x_2, \ldots) = \sum_{t=1}^{\infty} \alpha^{t-1} u(x_t) , \qquad 0 < \alpha < 1 ,$$

is maximized under given conditions of technology and population growth. Here x_t is per capita consumption in period t, and $u(x)$ is a strictly concave, increasing, single-period utility function. α is called a discount factor. If $\alpha = \frac{1}{1+\rho}$, then ρ is called a discount rate.

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**** See Ramsey [1928], Cass [1965], Koopmans [1965, 1967], Malinvaud [1965], and other papers cited there.

A generalization of (1) has been proposed under the name stationary utility,*

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* Koopmans [1960, 1966], Koopmans, Diamond and Williamson [1964].

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and is definable by a recursive relation

(2) $$U(x_1, x_2, x_3, \ldots) = V(x_1, U(x_2, x_3, \ldots)) .$$

One obtains (1) by $V(x, U) = u(x) + \alpha U$. The natural generalization of α in (1) to stationary utility is the function

(2a) $$\alpha(x) = \left(\frac{\partial V(x, U)}{\partial U}\right)_{U = U(x, x, x, \ldots)} .$$

In this paper we study the maximization of (2) under production assumptions, described below.

2. Definitions, notations and assumptions

We assume discrete time t , and a single commodity serving as capital (amount z_t at end of period t) and also as consumption good (flow x_t during period t) . Technology is constant and is represented by a production function $f(z)$. If the labor force is assumed constant, $f(z_t)$ represents output in period $t+1$, net of depreciation. If the labor force grows exponentially at a given rate $\lambda > 0$, z_t and x_t stand for capital and consumption per worker, and $f(z)$ represents output per worker less λz , the capital formation required in each period merely to keep z_t constant.

A <u>capital path</u> is a sequence $_0z = (z_0, z_1, \ldots)$, $0 \leqq z_t < \bar{z}$, where $0 < \bar{z} \leqq +\infty$. We denote by $_tz$ the tail $(z_t, z_{t+1}, \ldots)$ and by $_sz_t$ the finite segment $(z_s, z_{s+1}, \ldots, z_t)$.

A <u>consumption path</u> is a sequence, $x = (x_1, x_2, \ldots)$, $x_t \geqq 0$. We define the tail $_tx$ and the segment $_sx_t$ as above.

For any constant a , we denote by $_{con}a$ the constant (capital or consumption) path $(a, a, a, \ldots)$.

The capital path $_0z$ is said to be <u>feasible</u> for the initial capital stock z if $z_0 = z$ and

(3) $$z_{t+1} \leqq z_t + f(z_t) , \quad t = 0, 1, \ldots .$$

If $_0z$ is feasible for z the <u>associated consumption path</u> $_1x$ with

(4) $$x_{t+1} = z_t + f(z_t) - z_{t+1} \geqq 0 , \quad t = 0, 1, \ldots$$

is also said to be feasible for z . Let $\mathcal{Z}_z$ and $\mathcal{C}_z$ be the collections of capital paths and consumption paths, respectively, which are feasible for z .

We assume

(I) <u>The production function</u> $f(z)$ <u>is continuous and continuously differentiable on the interval</u> $\mathcal{I} = [0, \bar{z})$, $\bar{z} \leqq \infty$. <u>Moreover</u> $f(0) = 0$, $0 < f'(0)$, f <u>is concave, and the function</u> $h(z) = z + f(z)$ <u>is an increasing function mapping</u> $\mathcal{I}$ <u>onto itself.</u> <u>Hence</u> $h(\bar{z}) = \lim_{z \to \bar{z}} h(z) = \bar{z}$.

To interpret these assumptions, let $F(Z, L)$ represent total output before depreciation, Z the total capital stock, L the labor force.

The standard assumptions $F(0, L) = F(Z, 0) = 0$, $F'_L > 0$, $F'_Z > 0$, $F''_{ZZ} < 0$, F homogeneous of degree 1, then imply through $F(Z, L) = Lf(Z/L)$, ignoring depreciation, that $\bar{z} = \infty$. Either exponential labor force growth or a constant rate of depreciation will make $\bar{z}$ the finite number defined by $f(\bar{z}) = 0$. Should $z_0 > \bar{z}$, then feasibility requires $z_t < \bar{z} + \epsilon$ for any $\epsilon > 0$ and large enough t (see Figure 1). From assumptions on U made below we shall see that optimality requires $z_t < \bar{z}$ eventually. On the other hand, for $0 < z_0 < \bar{z}$, feasibility precludes $z_t \geqq \bar{z}$, whereas $z_0 = \bar{z}$ requires $z_t \leqq \bar{z}$. For these reasons we consider only values $z_0 \in \mathcal{J}$. We note for future use that if $0 < z'_0 < \bar{z}$, feasibility permits $\lim_{t \to \infty} z'_t = \bar{z}$; see Figure 1.

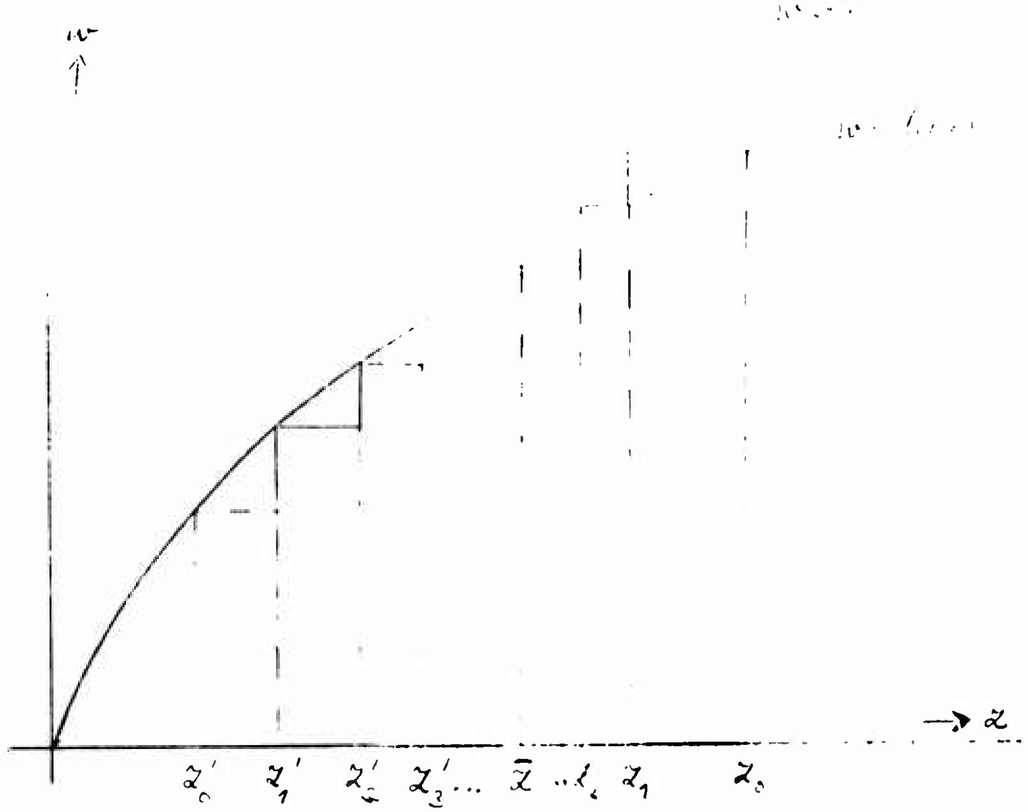

Figure 1. Two capital paths with zero consumption.

(II) $U({}_1x)$ is defined on the union $\mathcal{X} = \bigcup_{z \in \mathcal{J}} \mathcal{X}_z$ of all feasible sets, satisfies the recursive relation (2), and is continuous on each $\mathcal{X}_z$ with respect to the product topology.*

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* For a definition of the product topology see Kelley [1955], or use the distance function $D({}_1x, {}_1x') = \sum_{t=1}^{\infty} \delta^t \frac{|x_t - x'_t|}{1 + |x_t - x'_t|}$, where δ is any number with $0 < \delta < 1$.

An example where $U({}_1x)$ is continuous on each $\mathcal{X}_z$ but not on $\mathcal{X}$ is given below.

(III) $U({}_1x)$ is strictly quasi-concave on $\mathcal{X}$.

That is, ${}_1x(\lambda) = \lambda({}_1x) + (1-\lambda)({}_1x')$, $0 < \lambda < 1$, implies

$$U({}_1x(\lambda)) > \min\left\{U({}_1x),\ U({}_1x')\right\},$$

a standard assumption in utility theory. In general, it expresses a decreasing desire for one commodity or commodity bundle relative to another as the other is traded for the one at a constant barter ratio.

(IV) $V(x, U)$ has positive continuous derivatives $\partial V/\partial x$, $\partial V/\partial U$, on $\mathring{\mathcal{J}} \times \mathcal{U}$, where $\mathring{\mathcal{J}} = (0, \bar{z})$ and $\mathcal{U}$ is the range of $U({}_1x)$. Moreover $V(x, U)$ is continuous at $x = 0$ for all U, and, if V is not differentiable at $x = 0$, then $\lim_{x \to 0} \frac{\partial V(x, U)}{\partial x} = \infty$ for all U.

It follows from (II) and (IV) that $U({}_1x)$ strictly increases with each x_t.

The purpose of the exception at $x = 0$ is to permit a utility function for which " $z_o > 0$ " implies that " $\hat{x}_t > 0$ for all t ," where ${}_1\hat{x}$ denotes the optimal consumption path.

From the identity $U({}_{con}x) = V(x, U({}_{con}x))$ implied in (2) one finds by differentiation that (IV) implies $0 < \alpha(x) < 1$ for all $x > 0$ with ${}_{con}x \in \mathcal{X}$.

(5) (V) <u>Let</u> $V_2(x, y; U) = V(x, V(y, U))$ <u>and</u>

$$D(x, y; U) = -\left(\frac{dy}{dx}\right)_{V_2(x,\, y;\, U) \,=\, \text{const.}} = \frac{\partial V_2(x, y; U)}{\partial x} \Bigg/ \frac{\partial V_2(x, y; U)}{dy} .$$

<u>Then, for given</u> y, U, $D(x, y; U)$ <u>is strictly decreasing in</u> x <u>on</u> $\mathring{\mathcal{J}}$.

Together with an assumption we will not need, that $D(x, y; U)$ strictly increases with y , (V) is implied in the following plausible assumption: The first- and second-period consumptions $\hat{x}(B)$, $\hat{y}(B)$ that maximize $V_2(x, y; U)$ for given U if bought at given positive prices p, q within a budget $px + qy \leqq B$, are strictly increasing with B . Economically, consumption in neither period is inferior to that in the other period, in the way potatoes are inferior to steak.

The three assumptions just mentioned are illustrated in Figure 2.

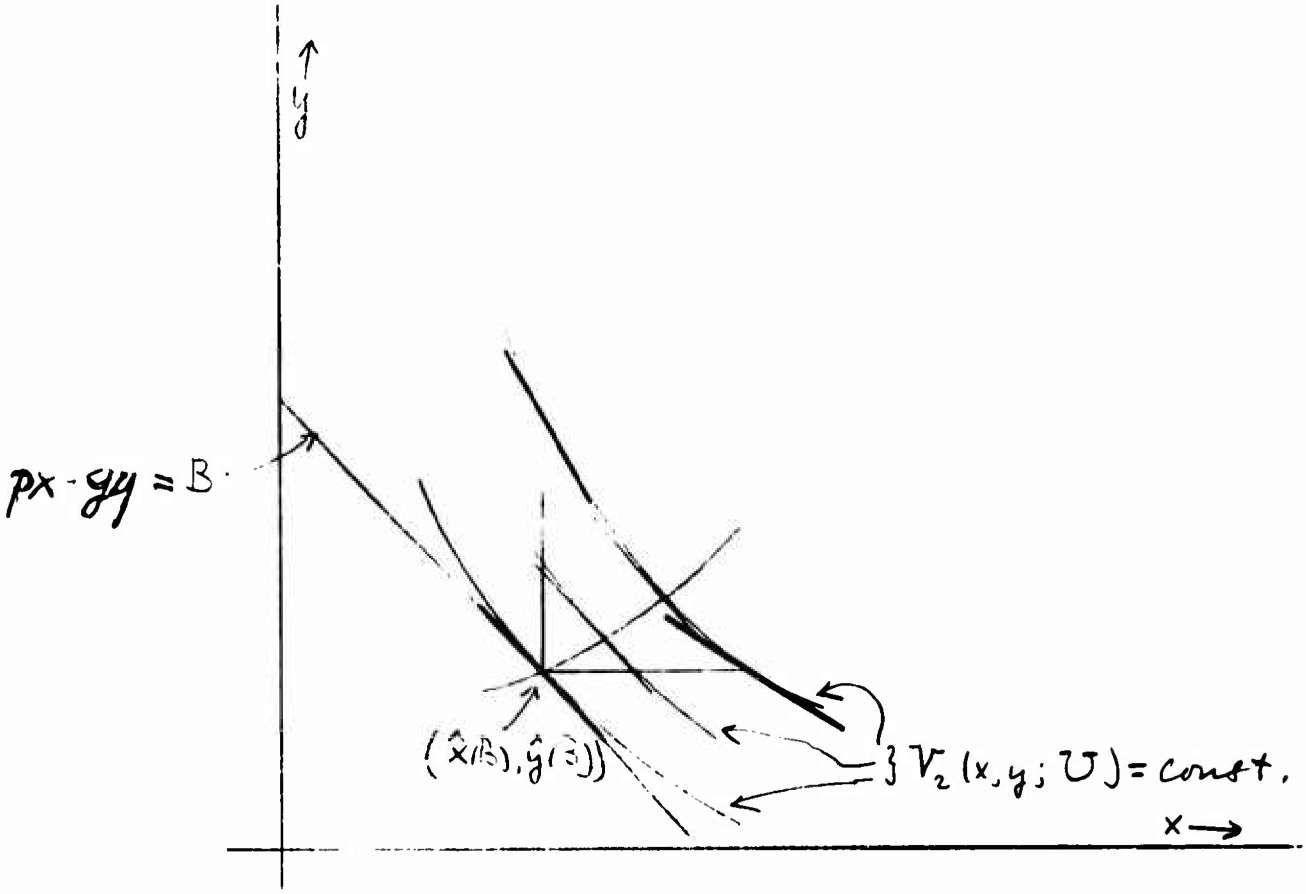

Figure 2. Noninferiority of consumption in Periods 1 and 2.

An example of a pair of functions $U({}_1x)$, $f(z)$, that satisfies all assumptions is given by (1) above, with $u(x) = x^{\gamma}$, $0 < \gamma < 1$, and any $f(z)$, concave and continuously differentiable on $\mathcal{X}' = [0, \infty)$ with $f(0) = 0$, $f'(0) > 0$, $\lim_{z \to \infty} f'(z) = 0$, hence $\lim_{z \to \infty} (f(z)/z) = 0$. Then, for any $\epsilon > 0$ and sufficiently large t , from (3), (4), $z_{t+1} \leqq h(z_t) \leqq (1+\epsilon)z_t$, hence $x_{t+1} \leqq h(z_t) \leqq (1+\epsilon)z_t \leqq (1+\epsilon)^{t-T} z_T$ for $t >$ some large T.

Taking $\epsilon < \alpha^{-1} - 1$ one sees that the summation (1) converges on each $\mathcal{X}_z$, hence on $\mathcal{X}$. Note that $U({}_1x)$ is not defined on all of $\mathcal{S}_x \mathcal{S}_x \mathcal{S}_x \ldots$, and is not continuous on $\mathcal{X}$ if $f(z)$ is not bounded; in fact, if $u(x) = x^{\frac{1}{2}}$, the sequence of consumption paths ${}_1x^{(n)}$ with $x_t^{(n)} = 0$, $t \neq n$ and $x_n^{(n)} = \alpha^{-2n}$ converges to ${}_{con}0$ in the product topology, but $U({}_1x^{(n)}) = 1$ for all n, whereas $U({}_{con}0) = 0$.

3. Optimal capital paths

Given a feasible capital path ${}_oz$, let ${}_1x$ be the associated consumption path given by (4). Define $W({}_oz)$ by $W({}_oz) = U({}_1x)$. If ${}_oz$ and ${}_oz'$ are in $\mathcal{Z}_{z_o}$, then the concavity of the production function $f(z)$ implies that a convex combination ${}_oz'' = \lambda({}_oz) + (1-\lambda)({}_oz')$, $0 < \lambda < 1$, is also in $\mathcal{Z}_{z_o}$, and that the associated consumption path ${}_o\mathbf{x}''$ has $x_t'' \geq \lambda x_t + (1-\lambda)x_t'$ for all t. This and the strict quasi-concavity of U imply that W is also strictly quasi-concave.

A capital path ${}_o\hat{z}$ is <u>optimal for</u> z if $\hat{z}_o \in \mathcal{Z}_z$, and $W({}_o\hat{z}) \geq W({}_oz)$ for all ${}_oz \in \mathcal{Z}_z$.

A capital path ${}_oz$ is <u>strictly monotone in time</u> if one of the following conditions holds:

(i) $z_t < z_{t+1}$, $t = 0, 1, 2, \ldots$;

(ii) $z_t = z_{t+1}$, $t = 0, 1, 2, \ldots$;

(iii) $z_t > z_{t+1}$, $t = 0, 1, 2, \ldots$;

$(\text{iii})_n$ $z_t > z_{t+1}$, $t < n$, $z_t = 0$, $t \geq n$.

The assumptions (I) - (V) in section 2 imply the following

Theorem 1. For any initial capital stock $z \in \mathcal{J}$ there is a unique optimal capital path ${}_0\hat{z}$. This path varies continuously with z and is strictly monotone in time.

If we define $h^{(n)}(z)$ recursively by $h^{(n)}(z) = h(h^{(n-1)}(z))$, $h^{(0)}(z) = z$, then the set $\mathcal{Z}_z$ is contained in the product $\mathcal{Z}_z^*$ of the closed intervals $[0, h^{(n)}(z)]$, $n = 0, 1, \ldots$. The latter set is compact with respect to the product topology, by the theorem of Tychonov, and $\mathcal{Z}_z$ is easily seen to be a closed subset, hence likewise compact. Continuity of U on $\mathcal{X}_z$ implies continuity of W on $\mathcal{Z}_z$. Then the continuous, strictly quasi-concave function W assumes a maximum at a unique element ${}_0\hat{z}$ of the compact convex set $\mathcal{Z}_z$. The remainder of this section is devoted to showing continuity and strict monotonicity of this unique optimal capital path ${}_0\hat{z}$.

Given $z \in \mathcal{J}$, let ${}_0\hat{z}$ be the optimal capital path for z and set $\hat{W}(z) = W({}_0\hat{z})$.

Lemma 1. $\hat{W}(z)$ is strictly increasing, and continuous from the left.

Proof. If $0 \leqq z < z' < \bar{z}$, and if ${}_0\hat{z}$ is optimal in $\mathcal{Z}_z$, let ${}_0z' \in \mathcal{Z}_{z'}$ be given by $z'_0 = z'$, ${}_1z' = {}_1\hat{z}$. Then for the associated consumption paths ${}_1x'$, ${}_1\hat{x}$, we have $x'_1 > \hat{x}_1$ and ${}_2x' = {}_2\hat{x}$, so $\hat{W}(z') \geqq W({}_0z') > W({}_0\hat{z}) = \hat{W}(z)$. Therefore $\hat{W}$ is increasing.

If $0 < z < \bar{z}$, then, in the optimal consumption path ${}_1\hat{x}$ associated with ${}_0\hat{z}$, some $\hat{x}_t$ is the first to be positive. Then $\hat{z}_{t'} > 0$ for $0 \leqq t' \leqq t - 1$, and for a sufficiently small $\epsilon > 0$ there is a $\delta > 0$ such that the path $({}_1\hat{x}_{t-1}, \hat{x}_t - \epsilon, {}_{t+1}\hat{x}) = {}_1x$ is feasible for $z - \delta$. Then $U({}_1x) \leqq \hat{W}(z-\delta) < \hat{W}(z)$. As $\delta \to 0$, $U({}_1x) \to \hat{W}(z)$, proving continuity from the left.

We can now show that ${}_0\hat{z}$ depends continuously on z. Suppose $z^{(n)} \to z \in \mathcal{I}$. For some $z' \in \mathcal{I}$, $z^{(n)} \leqq z'$ for all n. Then $\mathcal{Z}_{z^{(n)}} \subseteq \mathcal{Z}^*_{z'}$ for all n. Since the latter set is compact, it suffices to show that any convergent subsequence of the corresponding sequence of optimal paths, ${}_0\hat{z}^{(n)}$, must converge to ${}_0\hat{z}$, the optimal path for z. Renumbering, we may assume ${}_0\hat{z}^{(n)}$ itself converges to some ${}_0z \in \mathcal{Z}_z$. By the continuity of W, Lemma 1, and the optimality of ${}_0\hat{z}$ in $\mathcal{Z}_z$, respectively, $W({}_0z) = \lim W({}_0\hat{z}^{(n)}) = \lim \hat{W}(z^{(n)}) \geqq \hat{W}(z) = W({}_0\hat{z}) \geqq W({}_0z)$. Therefore $W({}_0z) = W({}_0\hat{z})$, so ${}_0z = {}_0\hat{z}$ by the uniqueness of ${}_0\hat{z}$, thus proving continuity of ${}_0\hat{z}$.

<u>Lemma 2.</u> <u>Suppose</u> $0 \leqq z < z' < \bar{z}$, <u>and let</u> ${}_0\hat{z}$ <u>and</u> ${}_0\hat{z}'$ <u>be the corresponding optimal paths</u>. <u>Then either</u> $\hat{z}_1 < \hat{z}_1'$ or $\hat{z}_1 = \hat{z}_1' = 0$.

<u>Proof</u>. Since $z = 0$ implies $\hat{z}_1 = 0$ the statement is obvious in that case.

Now assume $0 < z$. The stationarity of U (equation (2)) implies that for each t, ${}_t\hat{z}$ is optimal for $\hat{z}_t$. Therefore if $\hat{z}_1 = \hat{z}_1' \neq 0$, then ${}_1\hat{z} = {}_1\hat{z}'$. Suppose so, and let ${}_1\hat{x}$ and ${}_1\hat{x}'$ be the associated consumption paths. Then $\hat{x}_1 < \hat{x}_1'$ and $_2\hat{x} = {}_2\hat{x}'$. Write $U_3 = U(_3\hat{x}) = U(_3\hat{x}')$. Then $(\hat{x}_1, \hat{x}_2)$ maximizes $V_2(x, y; U_3)$ subject to $h(h(z) - x) - y = \hat{z}_2$, and similarly for $(\hat{x}_1', \hat{x}_2')$. But this is seen to contradict assumption (V), since $\hat{x}_1 < \hat{x}_1'$, $\hat{x}_2 = \hat{x}_2'$, and $h(z) - \hat{x}_1 = \hat{z}_1 = \hat{z}_1' = h(z') - \hat{x}_1'$, and in view of the concavity of h, the strict quasi-concavity of U, hence of V_2 (See Figure 3).

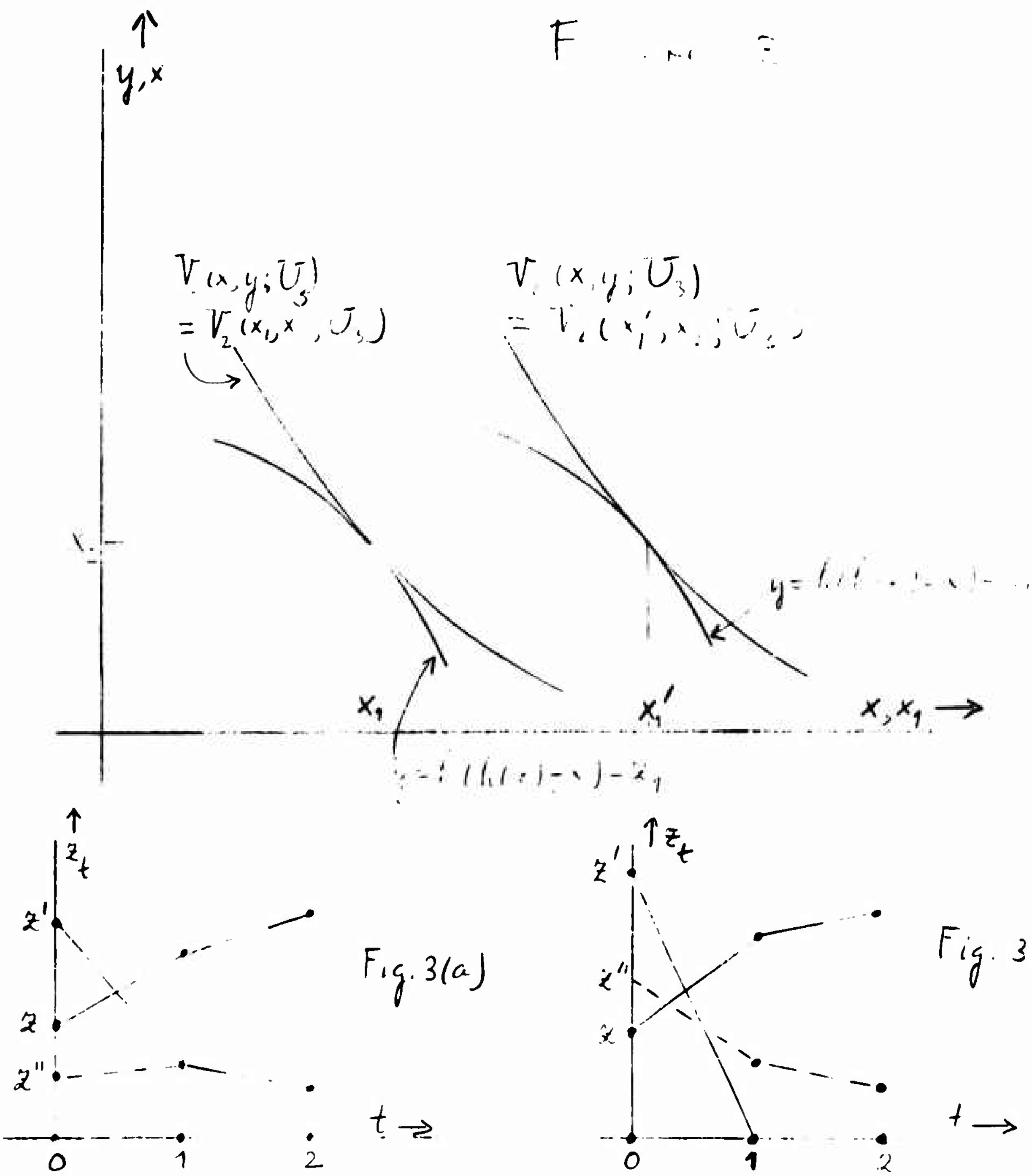

Fig. 3(a)

Fig. 3(b)

Now suppose $\hat{z}_1 > \hat{z}_1' > 0$. Moving from z toward zero and using continuity, we can find z'' with $0 < z'' < z'$ but with the corresponding $\hat{z}_1'' = \hat{z}_1'$; see figure 3(a). This was just shown to be impossible.

Finally, suppose $\hat{z}_1 > \hat{z}' = 0$. Moving from z' toward z we get a z'' with $z' > z'' > z$ and with the corresponding $\hat{z}_1''$ satisfying $0 < \hat{z}_1'' < \hat{z}_1$; see figure 3(b). But this is the case ruled out just above. This proves Lemma 2.

We now prove monotonicity of optimal capital paths. Suppose ${}_0\hat{z}$ is optimal for $z \in \mathcal{I}$, $z > 0$. Suppose first that $\hat{z}_0 < \hat{z}_1$. Now ${}_1\hat{z}$ is optimal for $\hat{z}_1$, so Lemma 2 implies $\hat{z}_1 < \hat{z}_2$. Inducing, we get $\hat{z}_t < \hat{z}_{t+1}$ for all t. The cases $\hat{z}_0 = \hat{z}_1$ and $\hat{z}_0 > \hat{z}_1$ are handled similarly.

4. Asymptotic behavior of optimal paths.

Monotonicity in time of the optimal path ${}_0\hat{z}$ implies that the (possibly infinite) limit $\hat{z}_\infty = \lim_{t\to\infty} \hat{z}_t$ exists. We want to determine, in terms of the initial capital stock z, when $\hat{z}_t$ increases, is constant, or decreases over time and what its limit is.

Suppose the pair $(\hat{x}, \hat{y})$ maximizes $V_2(x, y; U) = V(x, V(y, U))$ subject to the constraint $z_2 = h(h(z_0) - x) - y$, where U, z_0, and z_2 are given. Let $\hat{z}_1 = h(z_0) - \hat{x}$ and $\hat{U}_2 = V(\hat{y}, U)$. It follows from the usual analysis that, if $\hat{x} > 0$ and $\hat{y} > 0$, then

$$(6) \qquad \frac{\partial}{\partial x} V(\hat{x}, \hat{U}_2) = \frac{\partial}{\partial U} V(\hat{x}, \hat{U}_2) \cdot \frac{\partial}{\partial y} V(\hat{y}, U) \cdot (1 + f'(\hat{z}_1))$$

If $\hat{x}$ or $\hat{y}$ is zero, (6) is replaced by an appropriate inequality. Conversely, (6) or the corresponding inequality implies that $(\hat{x}, \hat{y})$ is optimal for the given problem.

Similarly ${}_1\hat{x}_n$ with each $\hat{x}_t > 0$ maximizes $V_n({}_1x_n, U) = V\Big(x_1, V(x_2, \ldots, V(x_n, U)\ldots)\Big)$ subject to ${}_1x_n$ being obtained by (4) from ${}_oz_n$ with z_o, z_n, U prescribed, if and only if

$$(7) \qquad \frac{\partial}{\partial x} V(\hat{x}_t, \hat{U}_{t+1}) = \frac{\partial}{\partial U} V(\hat{x}_t, \hat{U}_{t+1}) \cdot \frac{\partial}{\partial x} V(\hat{x}_{t+1}, \hat{U}_{t+2}) \cdot (1 + f'(z_t)) \; ,$$

$t = 1, 2, \ldots, n - 1$, where $\hat{U}_t = V_{n-t+1}({}_t\hat{x}_n, U)$ and $\hat{U}_{n+1} = U$.

A path ${}_oz$ with associated consumption path ${}_ox$ cannot be improved by finitely many changes in z_t, $t > 1$, if and only if the corresponding equations (7) hold for all t . Thus ${}_oz$ cannot be improved by finitely many changes if and only if it cannot be improved by a single change.

Given $z \in \mathcal{O}$, $z > 0$, the consumption path associated with ${}_{con}z$ is ${}_{con}x$, where $x = f(z)$. Let $U = U({}_{con}x)$. If ${}_{con}z$ were optimal we could divide (6) by $\frac{\partial}{\partial x} V(x, u)$ to get

$$(8) \qquad \alpha(f(z))(1 + f'(z)) = 1 \; ,$$

where $\alpha(x)$ is given by (2a).

Partition $[0, \bar{z}]$ into disjoint sets:

$$\mathcal{J}^{=} = \{ z \mid z = 0,\ z = \bar{z},\ \text{ or } \ \alpha(f(z))(1 + f'(z)) = 1 \},$$

$$\mathcal{J}^{>} = \{ 0 < z < \bar{z} \mid \alpha(f(z))\cdot(1 + f'(z)) > 1 \},$$

$$\mathcal{J}^{<} = \{ 0 < z < \bar{z} \mid \alpha(f(z))\cdot(1 + f'(z)) < 1 \}.$$

Then $\mathcal{J}^{=}$ is closed and $\mathcal{J}^{>}$, $\mathcal{J}^{<}$ are open. The preceding shows that a necessary condition for ${}_{con}z$ to be optimal if $z \in \mathcal{J}$ is that $z \in \mathcal{J}^{=}$. We shall show:

Theorem 2. Let ${}_{o}\hat{z}$ be optimal for z, $0 < z < \bar{z}$. Then

(a) if $z \in \mathcal{J}^{=}$, ${}_{o}\hat{z}$ is the constant path ${}_{con}z$;

(b) if $z \in \mathcal{J}^{>}$, then $\hat{z}_t$ increases and $\hat{z}_\infty$ is the smallest number in $\mathcal{J}^{=}$ which is larger than z;

(c) if $z \in \mathcal{J}^{<}$, then $\hat{z}_t$ decreases and $\hat{z}_\infty$ is the largest number in $\mathcal{J}^{=}$ which is smaller than z.

A path ${}_{o}\hat{z}$ optimal for z is called stable if for every path ${}_{o}\hat{z}'$ optimal for z' which has z' sufficiently near z, the limit $\hat{z}'_\infty = \hat{z}_\infty$. We have the following consequence of Theorem 2; see Figure 4.

Corollary. Let ${}_{o}\hat{z}$ be optimal for z. Then ${}_{o}\hat{z}$ is stable unless $z \in \mathcal{J}^{=}$ and is also in the closure of $\{z' \mid z' \in \mathcal{J}^{>},\ z' > z\}$ or of $\{z' \mid z' \in \mathcal{J}^{<},\ z' < z\}$.

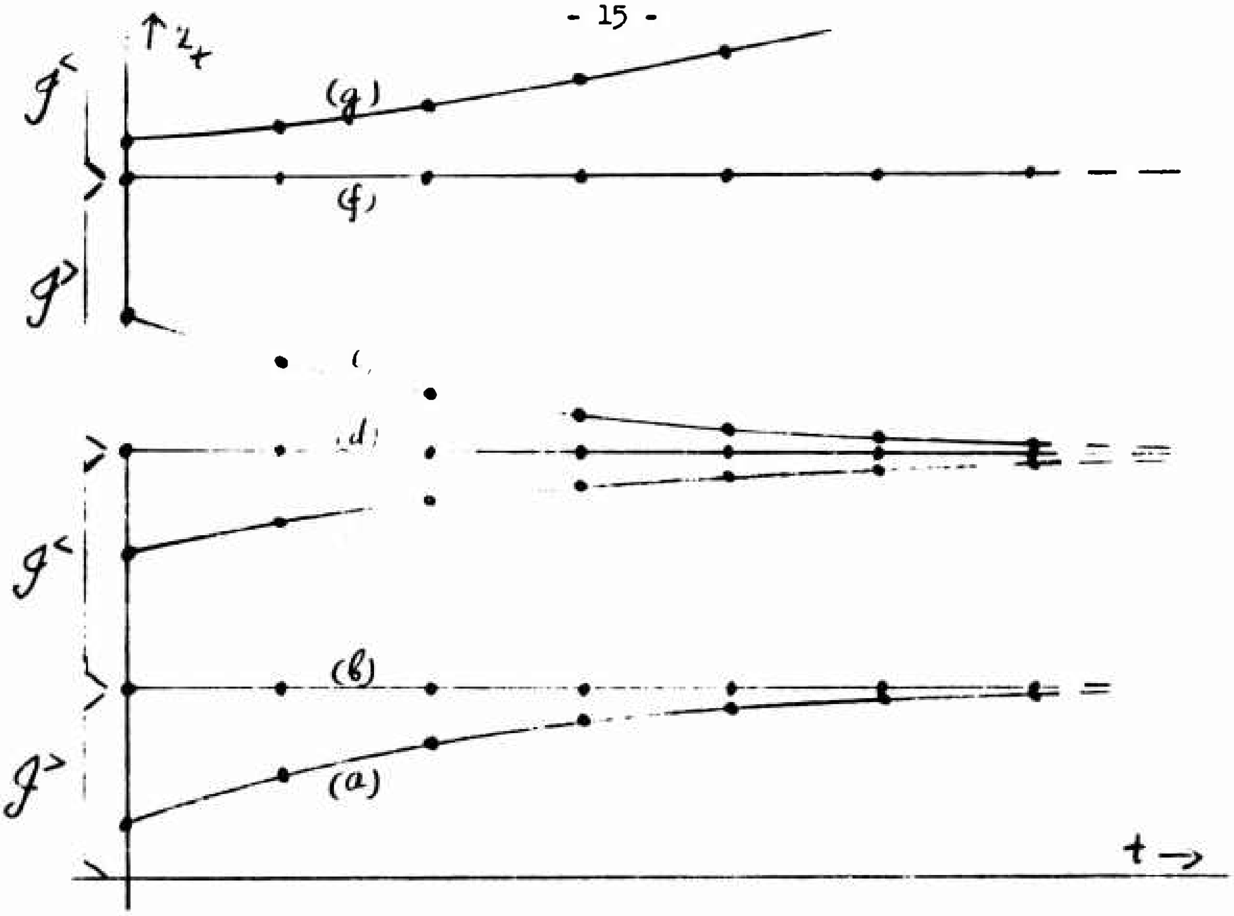

Figure 4. Optimal paths; all except (b) and (f) are stable.

If $z \in \mathcal{J}^{=}$, $0 < z < \bar{z}$, then (8) shows that the equations (7) are satisfied by the path ${}_{con}z = {}_{o}z$. Therefore ${}_{o}z$ cannot be improved by changing only finitely many of the z_t , $t \geqq 1$. Statement (a) of Theorem 2 is thus included in the following

Lemma 3. Let ${}_{o}z$ *be a feasible capital path with* $z_t \leqq z^* < \bar{z}$ *for all* t . *Suppose* $W({}_{o}z) \geqq W({}_{o}z')$ *for all* ${}_{o}z'$ *with* $z_o' = z_o$ *and* ${}_{m}z' = {}_{n}z$ *for some* m, n . *Then* ${}_{o}z$ *is optimal.*

<u>Proof</u>. Suppose ${}_0z'' \in \mathcal{Z}_{z_0}$, and suppose first that $z''_t > 0$, all t. For any n there is a path ${}_0z^{(n)} \in \mathcal{Z}_{z_0}$ with ${}_0z^{(n)}_n = {}_0z''_n$ and ${}_mz^{(n)} = {}_mz$ for sufficiently large m (depending on n, z''_n and ${}_0z$; this follows from the last remark preceding Figure 1 above. Then $W({}_0z) \geq W({}_0z^{(n)})$, and ${}_0z^{(n)} \to {}_0z''$ so $W({}_0z) \geq W({}_0z'')$. If z''_t is eventually zero, choose ${}_0z^{(n)}$ similarly but with $z^{(n)}_t = \max\{z''_t, \epsilon_n\}$ for $t \leq n$, where $\epsilon_n > 0$, $\epsilon_n \to 0$. Again we find $W({}_0z) \geq \lim W({}_0z^{(n)}) = W({}_0z'')$, so ${}_0z$ is optimal.

It is clear from the proof of Lemma 3 that the assumption that z_t is bounded away from $\bar{z}$ is stronger than necessary. What is needed is that ${}_tz$ can always be caught up with, even from a late and bad start. Some such assumption is clearly necessary, however, for let $z_t = h^{(t)}(z_0)$, all t. Then ${}_0z' \in \mathcal{Z}_{z_0}$ and ${}_nz' = {}_nz$ for some n implies ${}_0z' = {}_0z$. Thus ${}_0z$ cannot be improved by finitely many changes, since it cannot be changed in only finitely many places. However the associated consumption path is ${}_{con}0$, so ${}_0z$ is strictly inferior to any other path in $\mathcal{Z}_{z_0}$.

Next we consider the effect of finitely many changes in ${}_{con}z$ when $z \in \mathcal{O}^{>}$.

<u>Lemma 4.</u> <u>Suppose</u> $z \in \mathcal{O}^{>}$. <u>If</u> ${}_0z \in \mathcal{Z}_z$ <u>and</u> $z_t \leq z$ <u>for</u> $t < n$, <u>while</u> ${}_nz = {}_{con}z$, <u>then</u> $W({}_0z) \leq W({}_{con}z)$. <u>Moreover, equality holds only if</u> ${}_0z = {}_{con}z$.

<u>Proof</u>. We induce on n. By assumption $z_0 = z$, so for $n = 1$ there is nothing to prove. Suppose the statement is true for $n = m \geqq 1$ and suppose $z_t \leqq z$ for $t \leqq m$ while ${}_{m+1}z = {}_{con}z$. If $z_m = z$ then ${}_m z = {}_{con}z$ and the statement holds, by assumption. Suppose $z_m < z$. Choose a path ${}_0z' \in \mathcal{Z}_z$ with $z_t' = z$, $t \neq m$ and $z_m' = z + \delta$, $\delta > 0$. The corresponding value of W satisfies

$$W({}_0z') - W({}_{con}z) = \frac{\partial V}{\partial x}(x, U)[\alpha(x)(1 + f'(z)) - 1] \cdot \delta + \epsilon(\delta) \cdot \delta ,$$

where $x = f(z)$, $U = W({}_{con}z)$, and $\epsilon(\delta) \to 0$ as $\delta \to 0$. Since $z \in \mathcal{J}^>$, the factor in square brackets is positive. Therefore, for small positive δ, $W({}_0z') > W({}_{con}z)$. Now $z_m < z < z_m'$, so there is a convex combination ${}_0z''$ $= \lambda({}_0z) + (1-\lambda)({}_0z')$ with $z_m'' = z$. Clearly $z_t'' \leqq z$ for $t < m$ and ${}_m z'' = {}_{con}z$. The induction assumption implies that $W({}_0z'') \leqq W({}_{con}z)$. Strict quasi-concavity of W implies that $W({}_0z'') > \min\{W({}_0z), W({}_0z')\}$, but $W({}_0z') > W({}_{con}z) \geqq W({}_0z'')$, so $W({}_0z'') > W({}_0z)$. Therefore $W({}_{con}z) > W({}_0z)$, completing the proof.

A similar argument shows that if $z \in \mathcal{J}^<$, any change in ${}_{con}z$ moving finitely many z_t upward is a change for the worse.

We can now prove (b) of Theorem 2. Suppose $z \in \mathcal{J}^>$ and let ${}_0\hat{z}$ be the optimal path for z. We know that ${}_0\hat{z}$ is not constant, so it either increases or decreases. Suppose it decreased. As in the proof of Lemma 3, there would be a sequence of paths ${}_0z^{(n)} \in \mathcal{Z}_z$ such that ${}_0z^{(n)} \to {}_0\hat{z}$, $z_t^{(n)} \leqq z$ for all t, and ${}_m z^{(n)} = {}_{con}z$ for large m. By Lemma 4,

$W({}_{o}z^{(n)}) \leqq W({}_{con}z)$, all n . Therefore $W({}_{o}\hat{z}) \leqq W({}_{con}z)$,contradicting the unique optimality of ${}_{o}\hat{z}$, since ${}_{con}z$ in nonoptimal. Thus ${}_{o}\hat{z}$ increases.

Let z' be the smallest number in $\overset{=}{}$ which is larger than z . If $z' = \bar{z}$ then certainly $\hat{z}_{\infty} \leqq z'$. If $z' < \bar{z}$, then ${}_{con}z'$ is optimal for z' and repeated application of Lemma 2 shows that $\hat{z}_t < z'$ for all t . Thus again $\hat{z}_{\infty} \leqq z'$. Suppose $\hat{z}_{\infty} = z'' < \bar{z}$. Then ${}_{T}\hat{z}$ satisfies equations (7) for large T and all n , if we write $\hat{U}_{n+1} = W({}_{T+n-1}z)$. But ${}_{t}\hat{z} \to {}_{con}z''$. By continuity ${}_{con}z''$ will also satisfy equations (7) , with $\hat{U}_{n+1} = W({}_{con}z')$, so $z'' \in \overset{=}{}$. Then $z'' = z'$. This completes the proof of (b), and the proof of (c) is exactly parallel.

If ${}_{o}\hat{z}$ is an optimal capital path and ${}_{1}\hat{x}$ is the associated consumption path, then ${}_{1}\hat{x}$ obviously has the following properties:

$$\hat{x}_t \leqq f(z_t) \quad \text{if} \quad \hat{z}_t \quad \text{increases;}$$

$$\hat{x}_t \geqq f(z_t) \quad \text{if} \quad \hat{z}_t \quad \text{decreases;}$$

$$\hat{x}_{\infty} = \lim_{t\to\infty} \hat{x}_t = \lim_{t\to\infty} f(\hat{z}_t) .$$

It is not clear whether our assumptions guarantee that $\hat{x}_t$ is also monotone with respect to time. It is monotone when U has the special form (1), see equation (7)

5. Construction of optimal paths

We give two procedures for constructing the optimal capital path as a limit of a sequence of paths each obtained by solving the optimization problem for finite time. Each procedure has certain disadvantages, theoretical or practical.

Given a path ${}_0z \in \mathcal{Z}_z$ and an integer $n \geqq 1$, let $T_n({}_0z)$ be the path ${}_0z' \in \mathcal{Z}_z$ which maximizes $W({}_0z')$ with contraints ${}_0z'_{n-1} = {}_0z_{n-1}$, ${}_{n+1}z' = {}_{n+1}z$. Thus $T_n({}_0z)$ is obtained from ${}_0z$ by making the best feasible adjustment in z_n alone. Then T_n is an operator from $\mathcal{Z}_z$ to $\mathcal{Z}_z$. Note that $W(T_n({}_0z)) \geqq W({}_0z)$, with equality only when $T_n({}_0z) = {}_0z$.

Let S_n be the iterated operator $S_n = T_n T_{n-1} \cdots T_1$, and suppose $z \in \mathcal{J}$, $z > 0$. Start with some path ${}_0z^{(o)}$ in $\mathcal{Z}_z$ and define a sequence of paths inductively by

$$ {}_0z^{(n+1)} = S_{n+1}({}_0z^{(n)}) \; . $$

Thus ${}_0z^{(n+1)}$ is obtained by improving ${}_0z^{(n)}$ in the first $n+1$ places, in order. We cannot be sure that ${}_0z^{(n)}$ will converge to the optimal path ${}_0\hat{z}$; in fact if we make the unfortunate initial choice $z_t^{(o)} = h^{(t)}(z)$ for all t, then there is no room for finite change, so ${}_oz^{(n)} = {}_oz^{(o)}$ for all n, and ${}_oz^{(o)}$ is inferior to any ${}_oz \in \mathcal{Z}_z$.

Some subsequence ${}_oz^{(n,j)}$ will converge to a path ${}_oz \in \mathcal{Z}_z$. This path cannot be improved by a single change, so it cannot be improved by finitely many changes. In fact $W({}_oz^{(n)})$ is nondecreasing, and

$$W({}_oz^{(m)}) \leqq W(T_1({}_oz^{(m)})) \leqq W(S_{m+1}({}_oz^{(m)})) = W({}_oz^{(m+1)}) ,$$

so $W(T_1({}_oz)) = W({}_oz)$. Hence, by strict quasi-concavity of W, the adjustment of z_1 in the definition of $T_1({}_oz)$ leaves z_1 unchanged, and $T_1({}_oz) = {}_oz$. Inductively, suppose $T_j({}_oz) = {}_oz$ for $j \leqq n$. Then $T_{n+1}({}_oz) = S_{n+1}({}_oz)$, and the same argument shows $W(T_{n+1}({}_oz)) = W({}_oz)$, so $T_{n+1}({}_oz) = {}_oz$. If $z_t \leqq z^* < \bar{z}$ for all t, then, by Lemma 3, ${}_oz$ is optimal. Moreover, <u>if</u> ${}_oz$ <u>is optimal</u>, then any other convergent subsequence of ${}_oz^{(n)}$ will converge to a ${}_oz'$ with $W({}_oz') = W({}_oz)$. Hence the whole sequence ${}_oz^{(n)}$ will converge to ${}_oz$. As noted above, the limit need <u>not</u> be optimal, however.

An optimum can be guaranteed by the following method. Given $z \in \mathcal{J}$ with $z > 0$, choose some $z' \in \mathcal{J}$. (A computationally helpful choice of z' is the $\hat{z}_\infty$ of Theorem 2, provided $\hat{z}_\infty < \bar{z}$.) For some N there is a path ${}_oz' \in \mathcal{Z}_z$ with ${}_Nz' = {}_{con}z'$. For any $n \geqq N$ there is a unique ${}_oz^{(n)} \in \mathcal{Z}_z$ maximizing $W({}_oz)$ subject to ${}_{n}z = {}_{con}z'$. Let ${}_{o}\hat{z}$ be optimal for z . As in the proof of Lemma 3 there is a sequence of paths ${}_{o}\tilde{z}^{(n)} \to {}_{o}\hat{z}$ such that, for each n , the tail ${}_{m}\tilde{z}^{(n)}$ is eventually ${}_{con}z'$. Then $W({}_{o}\tilde{z}^{(n)}) \leqq W({}_{o}z^{(m)})$, so $\lim W({}_{o}z^{(m)}) = W({}_{o}\hat{z})$. It follows that ${}_{o}z^{(m)} \to {}_{o}\hat{z}$.

The practical difficulty with this method is that it involves solving optimization problems for more and more time periods, rather than for one period at each step as in the first method. Let us note that each such problem can be solved by iterating the one period solution. Suppose ${}_{o}z^{(o)} \in \mathfrak{Z}_z$ and $n \geqq 1$. A modification of the argument above shows that ${}_{o}z^{(m)} = (S_n)^m({}_{o}z^{(o)})$ converges to the path ${}_{o}z' \in \mathfrak{Z}_z$ which maximizes $W({}_{o}z')$ subject to ${}_{n+1}z' = {}_{n+1}z^{(o)}$.

REFERENCES

Cass, D. [1965], "Optimum Growth in an Aggregative Model of Capital Accumulation," Rev. of Econ. Stud., Vol. XXXII No. 3, 1965, pp. 233-240.

Koopmans, T. C. [1960], "Stationary Ordinal Utility and Impatience," Econometrica, April, 1960, pp. 287-309.

______, P. A. Diamond and R. E. Williamson [1964], "Stationary Utility and Time Perspective," Econometrica, January-April, 1964, pp. 82-100.

______ [1965], "On the Concept of Optimal Economic Growth," in The Econometric Approach to Development Planning, North Holland Publ. Co. and Rand McNally, 1966, a reissue of Pontificiae Academiae Scientiarum Scripta Varia, Vol. 28, 1965, pp. 225-300.

______ [1966], "Structure of Preference Over Time," Cowles Foundation Discussion Paper, No. 206, April 1966.

______ [1967], "Objectives, Constraints and Outcomes in Optimal Growth Models," Econometrica, January 1967.

Malinvaud, E. [1965], "Croissances optimales dans un modèle macroéconomique," in The Econometric Approach to Development Planning, North Holland Publ. Co., 1966, and Pontificiae Academiae Scientiarum Scripta Varia, 28, 1965, pp. 301-384.

Ramsey, F. P. [1928], "A Mathematical Theory of Saving," Economic Journal, December, 1928, pp. 543-559.